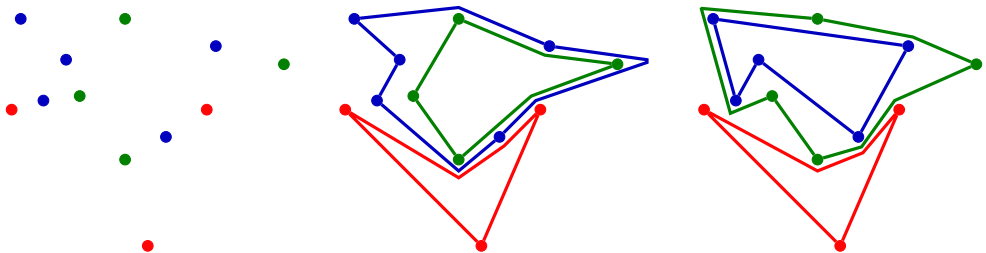
A $\left(\frac{5}{3}+\varepsilon\right)$-Approximation for Tricolored Non-Crossing Euclidean TSP

Júlia Baligács Yann Disser Andreas Emil Feldmann Anna Zych-Pawlewicz

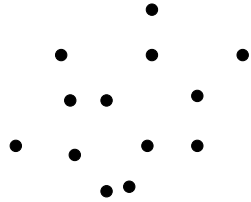
ESA 2024



Euclidean traveling salesperson problem

Euclidean traveling salesperson problem (ETSP)

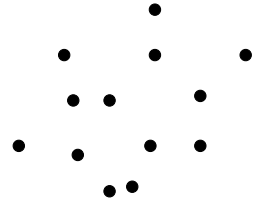
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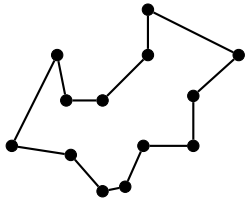


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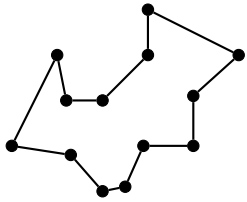


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complexity of ETSP

- NP-hard

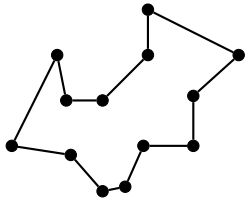
[Papadimitriou (TCS'1977)]

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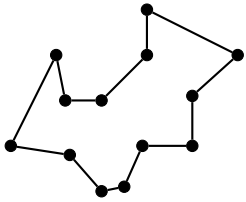
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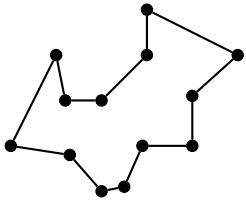
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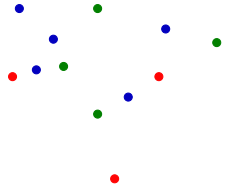
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  $\rightarrow$ dependency on ε optimal assuming ETH
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k-colored Euclidean TSP

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fixed: set C of k colors

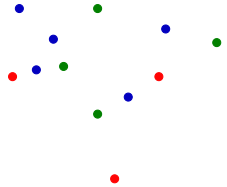


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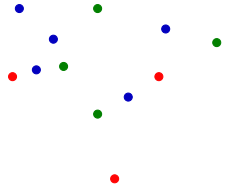
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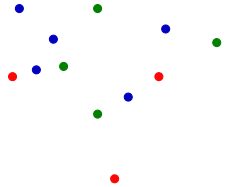
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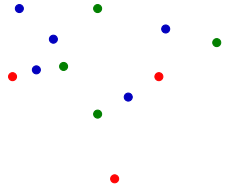
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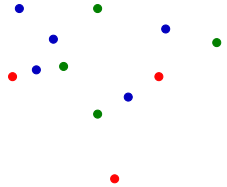
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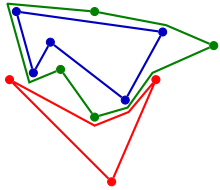
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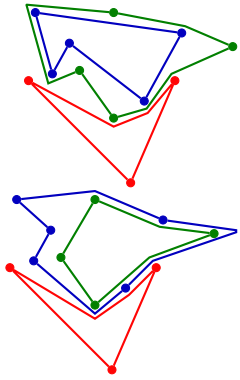
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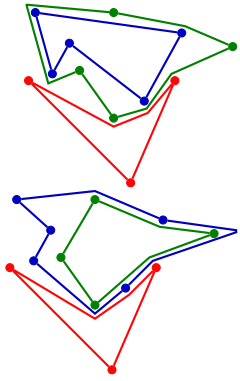
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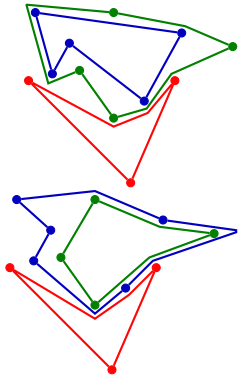
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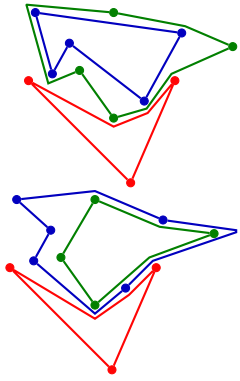
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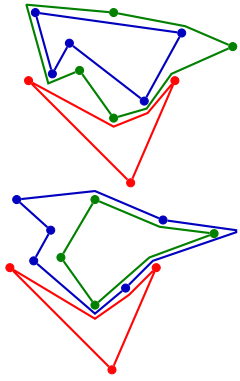
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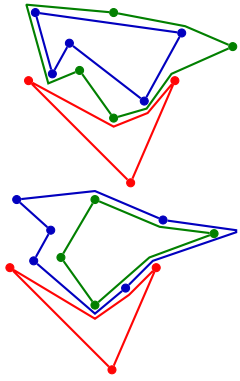
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Open problem. Does k-ETSP admit a PTAS for $k \geq 3$?

Main result

Theorem. For every $\varepsilon > 0$, there is an algorithm that computes a $\left(\frac{5}{3} + \varepsilon\right)$-approximation for 3-ETSP in time $\left(\frac{n}{\varepsilon}\right)^{O(1/\varepsilon)}$.

[B., Disser, Feldmann, Zych-Pawlewicz (2024)]

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naive idea: generalize Arora's algorithm (PTAS for ETSP)

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Theorem (Arora 1998). There is a PTAS for 1-ETSP.

i.e.: $\forall \varepsilon > 0$, compute $(1+\varepsilon)$-approximation in poly time

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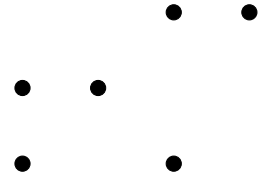
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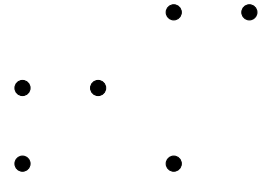
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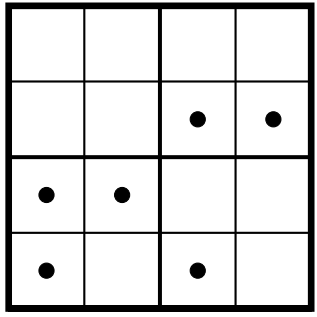
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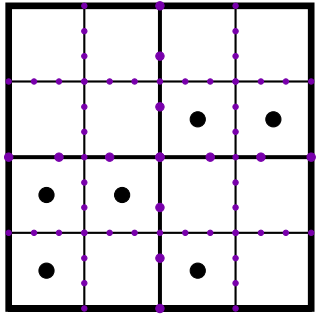
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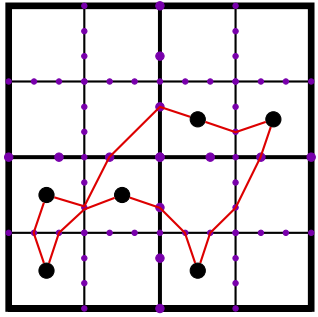
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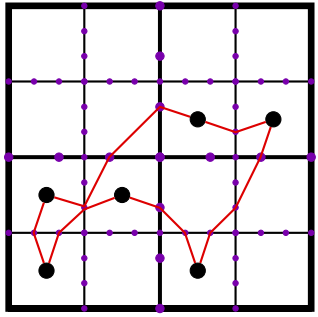
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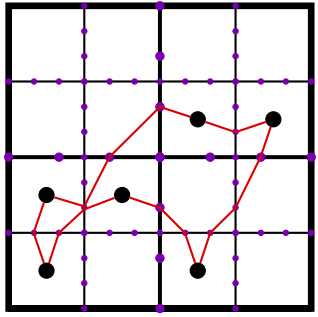
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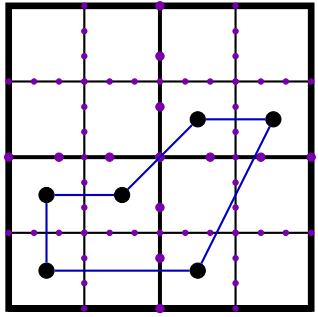
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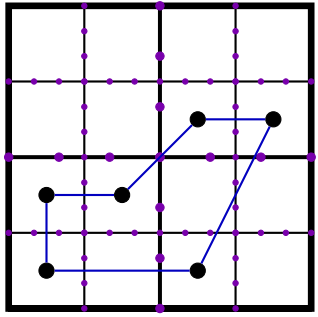
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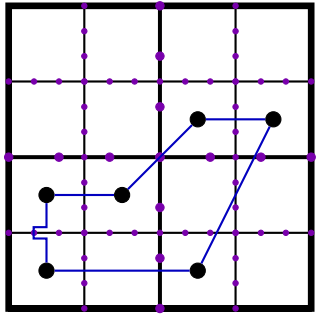
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Theorem (Arora 1998). There is a PTAS for 1-ETSP.

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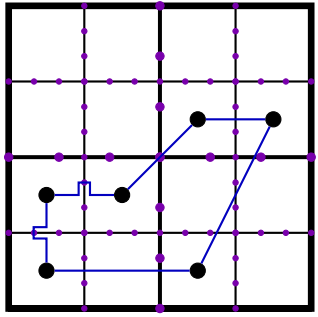
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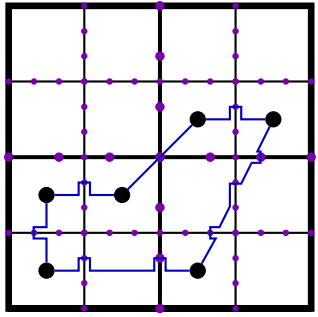
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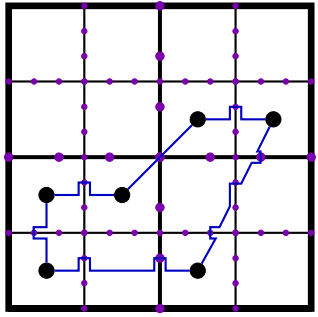
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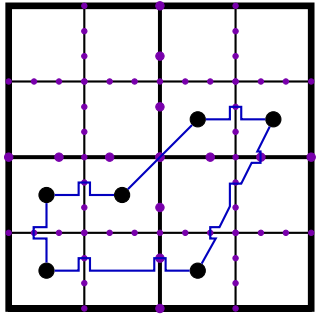
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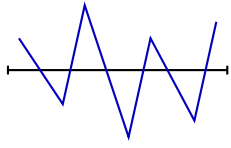
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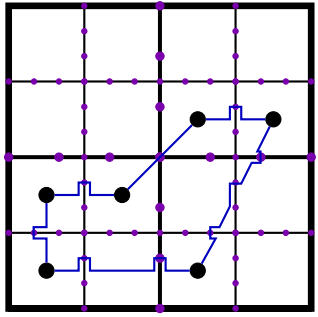


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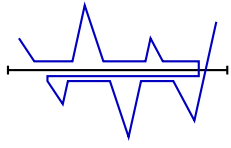
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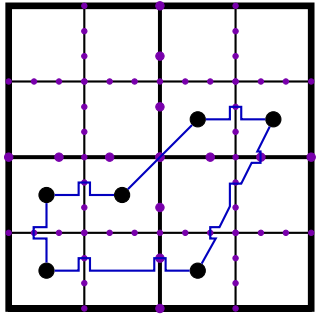


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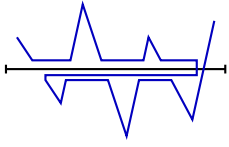
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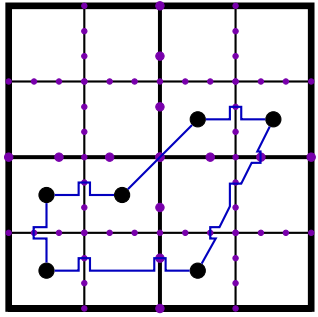


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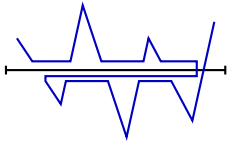
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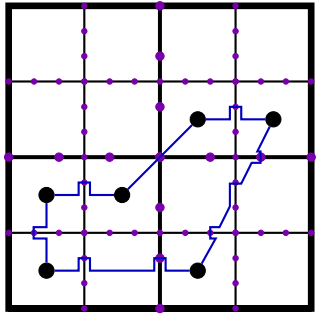


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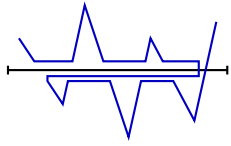
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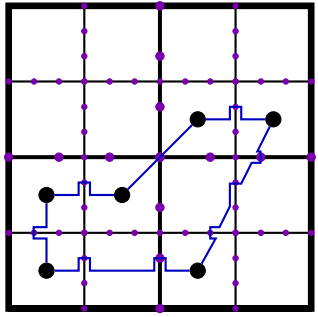


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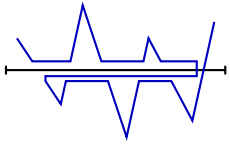
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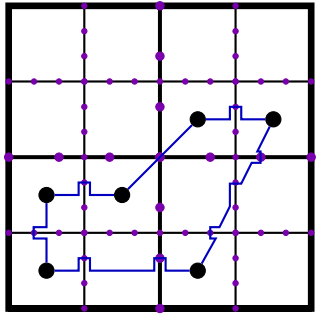
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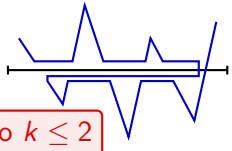
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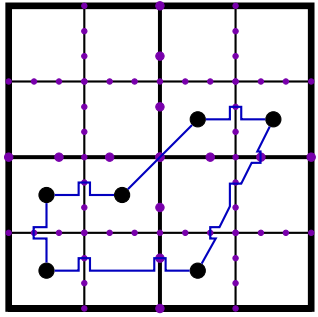


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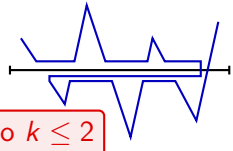
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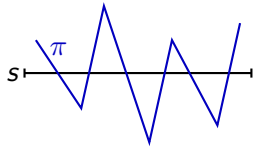
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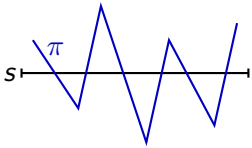
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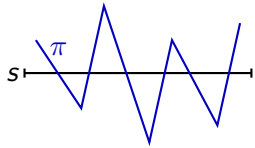
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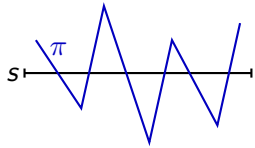
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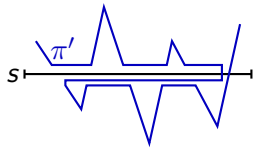
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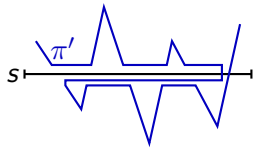
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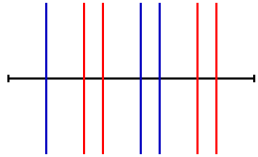


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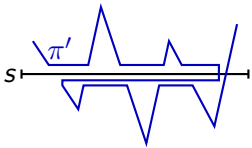
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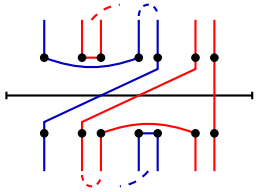
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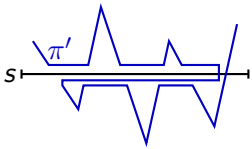
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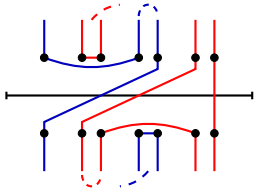
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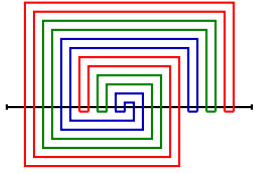
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three tours

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Conditional patching of three tours

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Then there are tours $\pi'_{\mathrm{R}}, \pi'_{\mathrm{B}}, \pi'_{\mathrm{G}}$ such that

- $\pi'_{\mathrm{R}}, \pi'_{\mathrm{B}}, \pi'_{\mathrm{G}}$ intersect s in at most 18 points,
- π_c and π'_c only differ in a small neighbourhood of s ($c \in \{\mathrm{R}, \mathrm{B}, \mathrm{G}\}$),
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Conditional patching of three tours

Tricolored Patching Lemma (B., Disser, Feldmann, Zych-Pawlewicz).
Let $\pi_{\mathrm{R}}, \pi_{\mathrm{B}}, \pi_{\mathrm{G}}$ be non-crossing tours and s a line segment such that **no red crossing is next to a green crossing** (along s).
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proof sketch

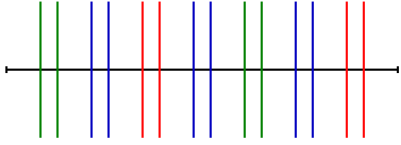
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proof sketch

- apply 1- and 2-color patching to reduce to the case



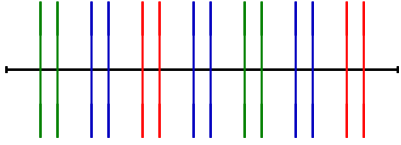
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proof sketch

- apply 1- and 2-color patching to reduce to the case
- apply patching scheme (up to symmetry)



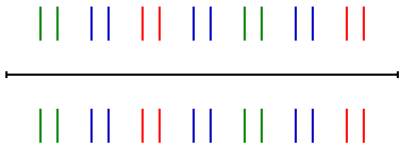
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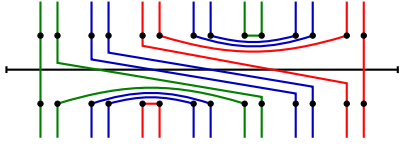
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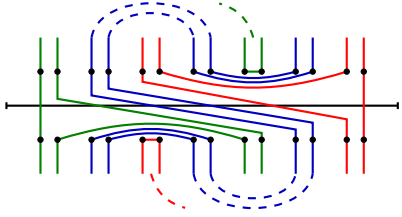
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proof sketch

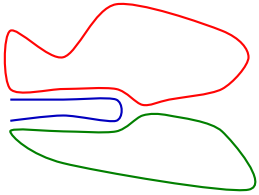
- apply 1- and 2-color patching to reduce to the case
- apply patching scheme (up to symmetry)
- argue that tours remain connected



Alternative approach

non-patchable case

- let $\pi_{\mathrm{R}}, \pi_{\mathrm{G}}, \pi_{\mathrm{B}}$ be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
- assume $\pi_{\mathrm{R}}, \pi_{\mathrm{G}} \leq \pi_{\mathrm{B}}$, then $l(\pi_{\mathrm{R}}) + l(\pi_{\mathrm{G}}) \leq \frac{2}{3} \cdot \mathrm{OPT}$



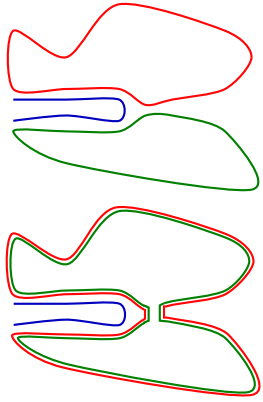
Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
- assume $\pi_R, \pi_G \leq \pi_B$, then $l(\pi_R) + l(\pi_G) \leq \frac{2}{3} \cdot \mathrm{OPT}$

approach

- connect and "double" red and green tour of OPT
  $\rightarrow$ solution of total length $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right) \cdot \mathrm{OPT}$



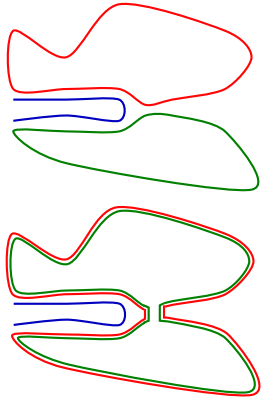
Alternative approach

non-patchable case

- let $\pi_{\mathrm{R}}, \pi_{\mathrm{G}}, \pi_{\mathrm{B}}$ be a $(1+\varepsilon)$ approximation
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approach

- connect and "double" red and green tour of OPT
  $\rightarrow$ solution of total length $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right) \cdot \mathrm{OPT}$
- merge R/G points, solve (weighted) 2-ETSP, "double" R/G-tour
  $\rightarrow$ gives $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right)$-approximation



Alternative approach

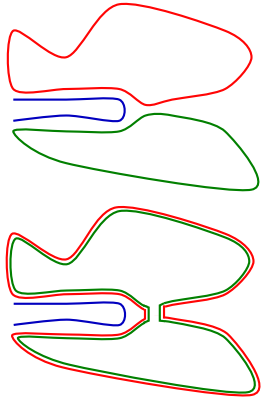
non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
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approach

- connect and "double" red and green tour of OPT
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final algorithm



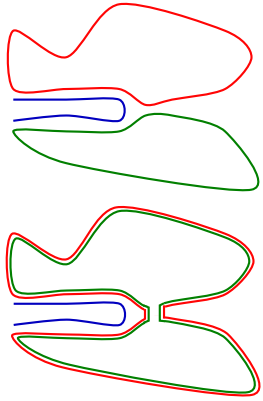
Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
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approach

- connect and "double" red and green tour of OPT
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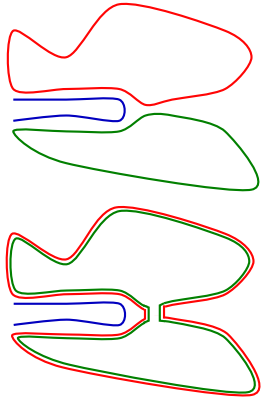
final algorithm

- compute optimal portal-respecting solution using generalized Arora's algorithm

Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
- assume $\pi_R, \pi_G \leq \pi_B$, then $l(\pi_R) + l(\pi_G) \leq \frac{2}{3} \cdot \mathrm{OPT}$



approach

- connect and "double" red and green tour of OPT
  $\rightarrow$ solution of total length $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right) \cdot \mathrm{OPT}$
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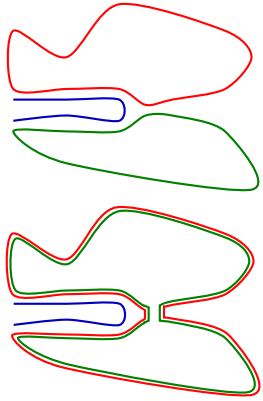
final algorithm

- compute optimal portal-respecting solution using generalized Arora's algorithm
- for every pair of colors, merge colors and use (weighted) PTAS for 2-ETSP

Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
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approach

- connect and "double" red and green tour of OPT
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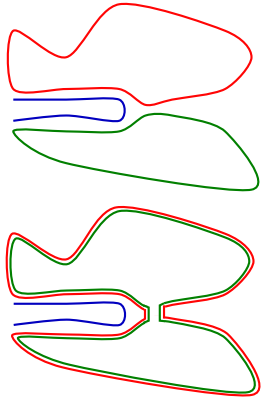
final algorithm

- compute optimal portal-respecting solution using generalized Arora's algorithm
- for every pair of colors, merge colors and use (weighted) PTAS for 2-ETSP
- output shortest solution

Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
- assume every pair of tours "meet"
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approach

- connect and "double" red and green tour of OPT
  $\rightarrow$ solution of total length $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right) \cdot \mathrm{OPT}$
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  $\rightarrow$ gives $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right)$-approximation

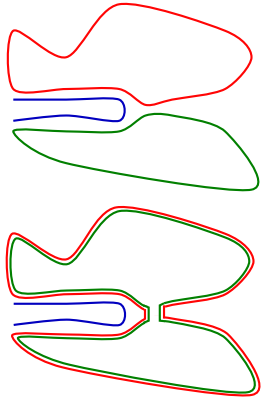
final algorithm

- compute optimal portal-respecting solution using generalized Arora's algorithm
  $\rightarrow$ gives $(1+\varepsilon)$-approximation if OPT is patchable
- for every pair of colors, merge colors and use (weighted) PTAS for 2-ETSP
- output shortest solution

Alternative approach

non-patchable case

- let π_R, π_G, π_B be a $(1+\varepsilon)$ approximation
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approach

- connect and "double" red and green tour of OPT
  $\rightarrow$ solution of total length $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right) \cdot \mathrm{OPT}$
- merge R/G points, solve (weighted) 2-ETSP, "double" R/G-tour
  $\rightarrow$ gives $\left(\frac{5}{3} + \mathcal{O}(\varepsilon)\right)$-approximation

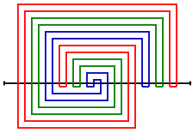
final algorithm

- compute optimal portal-respecting solution using generalized Arora's algorithm
  $\rightarrow$ gives $(1+\varepsilon)$-approximation if OPT is patchable
- for every pair of colors, merge colors and use (weighted) PTAS for 2-ETSP
  $\rightarrow$ gives $(5/3+\varepsilon)$-approximation if OPT is non-patchable
- output shortest solution

Summary

$\left(\frac{5}{3}+\varepsilon\right)$-approximation for 3-ETSP

main issue:
patching fails for $k \geq 3$

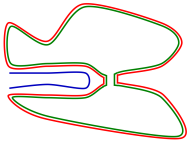


optimum patchable?

no

yes

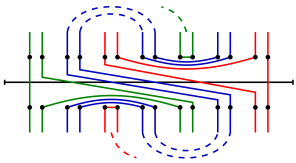
alternative approach
via merging colors



generalized Arora

existence of good
portal-resp. solution

computing optimal
portal-resp. solution



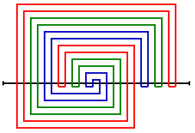
conditional patching
for 3 tours

generalized
dynamic program

Summary

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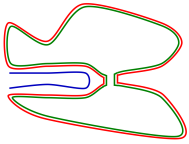


optimum patchable?

no

yes

alternative approach
via merging colors



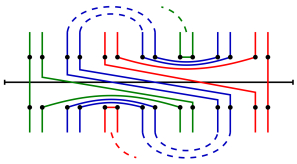
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Thank you!